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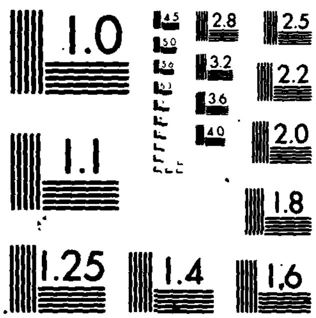

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General stochastic integral systems involving McShane integrals are studied. These systems include classes of stochastic integro-differential systems. Existence, uniqueness, and stability of solutions to such systems are investigated using contractor theory. In particular, the results yield frequency-type conditions for the stability of a class of stochastic systems.

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ON THE SOLUTION OF A CLASS OF STOCHASTIC
INTEGRAL SYSTEMS

by

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Abstract

General stochastic integral systems involving McShane integrals are studied. These systems include classes of stochastic integro-differential systems. Existence, uniqueness, and stability of solutions to such systems are investigated using contractor theory. In particular, the results yield frequency-type conditions for the stability of a class of stochastic systems.

1. Introduction

Stochastic equations are important in stochastic systems theory as well as in other areas of science and engineering [9]. Existence, uniqueness, stability, and approximation of solutions to such equations have been studied by several authors, for example, Bharucha-Reid [2], Blankenship [3], Lee and Padgett [4], Padgett and Rao [6], Padgett and Tsokos[7] , to name a few. In this paper, a general stochastic integral system which includes classes of stochastic integro-differential systems is studied. In particular, this paper generalizes the work of Padgett and Tsokos [7] and of Rao and Tsokos [8] on integro-differential systems and yields frequency-type conditions for the stability of a class of stochastic systems, extending the work of Blankenship [3].

Consider the stochastic integral system

$$
\begin{aligned}
x_i(t;\omega) &= x_{i,0}(\omega) + \int_0^t h_i(s,x(s;\omega);\omega)ds \\
&+ \sum_{j=1}^{p} \int_0^t r_{1ij}(t,\tau) \int_0^\tau K_{1ij}(\tau,s;\omega) f_{1ij}(s,x(s;\omega))dz_j(s;\omega)d\tau \\
&+ \sum_{j,\ell=1}^{p} \int_0^t r_{2ij\ell}(t,\tau) \int_0^\tau K_{2ij\ell}(\tau,s;\omega) f_{2ij\ell}(s,x(s;\omega))dz_j(s;\omega)dz_\ell(s;\omega)d\tau, \\
& \qquad i=1,2,\ldots,n, \qquad\qquad (1.1)
\end{aligned}
$$

where

(i) $t \in R^+ \equiv [0,\infty)$, $\omega \in \Omega$, the supporting set of a complete probability measure space (Ω,A,P);

(ii) $h_i\colon R^+ \times R^n \times \Omega \to R \equiv (-\infty,\infty)$;

(iii) $r_{1ij}(t,\tau)$, $r_{2ij\ell}(t,\tau)$ are continuous real-valued functions defined for $0 \le \tau \le t < \infty$;

(iv) $K_{1ij}(t,s;\omega)$ and $K_{2ij\ell}(t,s;\omega)$ are real-valued functions defined for $0 \le s \le t < \infty$ and $\omega \in \Omega$;

(v) $f_{1ij}, f_{2ij\ell}: R^+ \times R^n \to R$;

(vi) $z(t;\omega)$ with subscript is a real-valued stochastic process satisfying certain conditions; and

(vii) $x(t;\omega)$ is a vector stochastic process with components $x_i(t;\omega)$, $i = 1,2,\ldots,n$.

The integrals involving the process $z(s;\omega)$ in the equation (1.1) are to be understood as McShane integrals [5]. Throughout the paper we consider sums involving subscripts i, j and ℓ. For notational simplicity, we let

$\sum_i \equiv \sum_{i=1}^{n}$ and $\sum_{j,\ell} \equiv \sum_{j,\ell=1}^{p}$ unless otherwise stated.

2. Preliminaries

Let (Ω,A,P) be a complete probability measure space. We shall assume that there is a family of sub-σ-algebras $A_t, t \in R^+$, such that for $s < t$, $A_s \subset A_t$. We shall further assume with respect to equation (1.1) that

(H_1) every process denoted by $z(t;\omega)$ with subscript is a real-valued stochastic process adapted to A_t, with almost surely (a.s.) continuous sample functions, and satisfies the condition $E[(z(t;\omega) - z(s;\omega))^r | A_s] \le K(t-s)$, where $0 \le s \le t < \infty$, $r = 1,2,4$ and K is some constant;

(H_2) $x_0(\omega)$ is measurable with respect to A and is mean-square continuous;

(H_3) $K_{1ij}(t,s;\omega)$ and $K_{2ij\ell}(t,s;\omega)$ are adapted to A_s for all $t \ge s$ and are continuous as maps from $\Delta = \{(t,s): 0 \le s \le t < \infty\}$ into L_∞,

the set of all P-essentially bounded random variables (define the norm $|||K(t,s;\omega)||| = P - \text{ess}\sup_{\omega\in\Omega}|K(t,s;\omega)|$); and

(H_4) for any n-dimensional mean-square continuous process $x(t;\omega)$ adapted to A_t, $h(t,x(t;\omega);\omega)$, $f_{1ij}(t,x(t;\omega))$ and $f_{2ij\ell}(t,x(t;\omega))$ are adapted to A_t and are mean-square continuous.

Under the above assumptions, it is known [5,pp. 61-70, 138-139] that for an n-dimensional mean-square continuous process $x(s;\omega)$ adapted to A_s, the McShane integrals in equation (1.1) exist, are adapted to A_t, and satisfy the following inequalities:

$$[E\{\int_0^t K_{1ij}(t,s;\omega)f_{1ij}(s,x(s;\omega))dz_j(s;\omega)\}^2]^{\frac{1}{2}}$$

$$\leq 2K\int_0^t [E\{K_{1ij}(t,s;\omega)f_{1ij}(s,x(s;\omega))\}^2]^{\frac{1}{2}}ds$$

$$+ E\{\int_0^t K_{1ij}(t,s;\omega)f_{1ij}(s,x(s;\omega))dz_j(s,\omega)\}^2$$

$$\leq c^2\int_0^t E\{K_{1ij}(t,s;\omega)f_{1ij}(s,x(s;\omega))\}^2 ds \tag{2.1}$$

and

$$E\{\int_0^t K_{2ij\ell}(t,s;\omega)f_{2ij\ell}(s,x(s;\omega))dz_j(s;\omega)dz_\ell(s;\omega)\}^2$$

$$\leq c^2\int_0^t E\{K_{2ij\ell}(t,s;\omega)f_{2ij\ell}(s,x(s;\omega))\}^2 ds \tag{2.2}$$

where $c = 2K\sqrt{t} + \sqrt{K}$, K is the constant defined in (H_1), and E denotes expectation.

We shall now define some specific function spaces that will be used in this study. Let $y(t;\omega)$ be a second-order scalar process adapted to A_t. Denote

$$||y(t;\omega)||_2 = [E\{y(t;\omega)\}^2]^{\frac{1}{2}}. \tag{2.3}$$

Definition 2.1. $C \equiv C(R^+, L_2(\Omega,A,P))$ will denote the space of scalar mean-square continuous functions $y(t;\omega)$ adapted to A_t. We shall induce a topology on C by the family of semi-norms

$$||y(t;\omega)||_{C_n} = \sup_{t\in[0,n]} ||y(t;\omega)||_2, n = 1,2,\dots \quad . \tag{2.4}$$

It is known [9] that this topology is metrizable and the resulting metric space is a Fréchet space.

Definition 2.2. C_1 will denote the set of functions $y(t;\omega)$ in C such that

$$\sup_{t\geq 0} ||y(t;\omega)||_2 < \infty.$$

Then, C_1 is a Banach space with norm $||\cdot||_{C_1}$ defined by

$$||y(t;\omega)||_{C_1} = \sup_{t\geq 0} ||y(t;\omega)||_2.$$

Definition 2.3. Let B and D be Banach spaces in C and let T be an operator on C. Then the pair (B,D) is said to be admissible with respect to T if $TB \subset D$.

Definition 2.4. A Banach space B in C is said to be stronger than C if every sequence which converges in the norm of B also converges in the topology of C.

Definition 2.5. C_∞ will denote the space of continuous real functions $u(t)(t\geq 0)$ with norm $||u||_\infty$ defined by

$$||u||_\infty = \sup_{t\geq 0} |u(t)|.$$

Definition 2.6. Let $x(t;\omega)$ be a vector process with components $x_i(t;\omega)(i = 1,2,\dots,n)$. Then $x(t;\omega) \in C^n$ (B^n or D^n) if and only if $x_i(t;\omega) \in C(B$ or $D)$.

The following lemma is well-known [9].

Lemma 2.1. Let T be a continuous linear operator on C. Let B and D be Banach spaces stronger than C such that (B,D) is admissible with respect to T. Then T is continuous from B to D.

We shall now introduce the concept of a bounded integral vector contractor for a set of vector valued functions. Let B and D be Banach spaces stronger than C. Define the linear operators T, T_{ij}, and $T_{ij\ell}$ on C by

$$(Ty)(t;\omega) = \int_0^t y(s;\omega)ds, \tag{2.5}$$

$$(T_{ij}y)(t;\omega) = \int_0^t r_{1ij}(t,\tau) \int_0^\tau K_{1ij}(\tau,s;\omega)y(s;\omega)dz_j(s;\omega)d\tau \tag{2.6}$$

$$(T_{ij\ell}y)(t;\omega) = \int_0^t r_{2ij\ell}(t,\tau) \int_0^\tau K_{2ij\ell}(\tau,s;\omega)y(s;\omega)dz_j(s;\omega)dz_\ell(s;\omega)d\tau. \tag{2.7}$$

Assume that the pair (B,D) is admissible with respect to each of the operators T, T_{ij}, and $T_{ij\ell}$. Let $h(t,x;\omega)$, $f_{1ij}(t,x;\omega)$, and $f_{2ij\ell}(t,x;\omega)$ be real-valued functions such that $h(t,x(t;\omega);\omega)$, $f_{1ij}(t,x(t;\omega);\omega)$, $f_{2ij\ell}(t,x(t;\omega);\omega)$ are in B whenever $x(t;\omega) \in D^n$. Let f_1 denote the $n \times p$ matrix with elements f_{1ij} and let f_2 denote the $n \times p \times p$ three-dimensional matrix with elements $f_{2ij\ell}$.

Definition 2.7. The set of functions (h,f_1,f_2) is said to have a bounded integral vector contractor $(\Gamma,\Gamma_1,\Gamma_2)$ with respect to (B^n,D^n) if

(1) $\Gamma \equiv \Gamma(t,x)$ is a bounded linear operator from D to B for each $t \in R^+$ and $x \in R^n$. The function $||\Gamma(t,x)||$ is continuous in (t,x) and $||\Gamma(t,x)|| \leq Q(t)$ where $Q(t)$ is a bounded continuous function;

(2) Γ_2 is an $(n \times p)$ matrix of operators $\Gamma_{1ij}(t,x)$ such that for each $t \in R^+$ and $x \in R^n$, $\Gamma_{1ij}(t,x)$ is a bounded linear operator from D to B. The function $||\Gamma_{1ij}(t,x)||$ is continuous in (t,x) and

$\sum_{i,j} ||\Gamma_{1ij}(t,x)|| \le Q_1(t)$, where $Q_1(t)$ is bounded and continuous;

(3) Γ_2 is a three-dimensional matrix of operators $\Gamma_{2ij\ell}(t,x)$ such that for each $t \in R^+$, $x \in R^n$, $\Gamma_{2ij\ell}(t,x)$ is a bounded linear operator from D to B. The function $||\Gamma_{2ij\ell}(t,x)||$ is continuous in (t,x) and $\sum_{i,j,\ell} ||\Gamma_{2ij\ell}(t,x)|| \le Q_2(t)$ where $Q_2(t)$ is bounded and continuous; and

(4) for $x(t;\omega)$, $y(t;\omega) \in D^n$, the following inequalities hold:

$$\begin{aligned} ||h(t,x(t;\omega) &+ y(t;\omega) + (T\Gamma y)(t;\omega) \\ &+ ((\sum_j T_{1ij}\Gamma_{1ij})y)(t;\omega) \\ &+ ((\sum_{j,\ell} T_{2ij\ell}\Gamma_{2ij\ell})y)(t;\omega);\omega) \\ &- h(t,x(t;\omega);\omega) - (\Gamma y_i)(t;\omega)||_B \le \alpha_i ||y||_{D^n}; \end{aligned}$$

$$\begin{aligned} ||f_{1ij}(t,x(t;\omega) &+ y(t;\omega) + (T\Gamma y)(t;\omega) \\ &+ ((\sum_j T_{1ij}\Gamma_{1ij})y)(t;\omega) \\ &+ ((\sum_{j,\ell} T_{2ij\ell}\Gamma_{2ij\ell})y)(t;\omega)) \\ &- f_{1ij}(t,x(t;\omega)) - (\Gamma_{1ij}y_i)(t;\omega)||_B \le \alpha_{1ij} ||y||_{D^n}; \end{aligned}$$

$$\begin{aligned} ||f_{2ij\ell}(t,x(t;\omega) &+ y(t;\omega) + (T\Gamma y)(t;\omega) \\ &+ (\sum_j T_{1ij}\Gamma_{1ij}y)(t;\omega) \\ &+ (\sum_{j,\ell} T_{2ij\ell}\Gamma_{2ij\ell}y)(t;\omega)) \\ &- f_{2ij\ell}(t,x(t;\omega)) - (\Gamma_{2ij\ell}y_i)(t;\omega)||_B \le \alpha_{2ij\ell} ||y||_{D^n}, \end{aligned}$$

where $T\Gamma y$, $(\sum_j T_{1ij}\Gamma_{1ij})y$, and $(\sum_{j,\ell} T_{2ij\ell}\Gamma_{2ij\ell})y$ are n-vectors with components $T\Gamma y_i$, $(\sum_j T_{1ij}\Gamma_{1ij})y_i$, and $(\sum_{j,\ell} T_{2ij\ell}\Gamma_{2ij\ell})y_i$, respectively. Let α denote n-vector with components α_i, α_1 the $n\times p$ matrix of constants α_{1ij} and α_2, the three-dimensional $n\times p\times p$ matrix of constants $\alpha_{2ij\ell}$. Then the triplet $(\alpha,\alpha_1,\alpha_2)$ will be called the <u>vector of contractor constants</u>.

We remark that from equation (2.5) and the assumption (H_3) on K_1 and K_2 and from the inequalities (2.1) and (2.2), it is readily demonstrated that the operator T, T_{1ij}, and $T_{2ij\ell}$ are continuous on C. Hence, if a pair of spaces (B,D) are both stronger than C and admissible with respect to T, T_{1ij} and $T_{2ij\ell}$, it follows from Lemma 2.1 that these are bounded linear operators from B to D. Therefore, there are constants k, k_{1ij} and $k_{2ij\ell}$ such that

$$||T|| \le k, \tag{2.8}$$

$$||T_{1ij}|| \le k_{1ij}, \quad \text{and} \tag{2.9}$$

$$||T_{2ij\ell}|| \le k_{2ij\ell} \tag{2.10}$$

<u>Definition 2.8</u>. The random function $x(t;\omega)$ will be called a solution of the equation (1.1) if $x(t;\omega) \in C^n$ and satisfies (1.1) a.s.

3. <u>Existence of Solutions</u>

In this section the concept of a bounded integral vector contractor, as defined in the previous section, will be used to obtain the existence and uniqueness of solutions of the general stochastic integral system (1.1). The conditions under which the existence and uniqueness will be proven are

very general, and as a specific application, Theorem 3.3 will be used in Section 4 to study stability of stochastic systems under frequency-type conditions.

In the following theorems, B and D will denote Banach spaces stronger than C and B^n and D^n will be the corresponding product spaces.

Theorem 3.1 Let the system (1.1) satisfy the following conditions:

(1) $x_0(\omega) \in D^n$;

(2) the pair (B,D) is admissible with respect to each of the operators T, T_{1ij} and $T_{2ij\ell}$;

(3) the set of functions (h,f_1,f_2) has a bounded integral vector contractor $(\Gamma,\Gamma_1,\Gamma_2)$ with the vector of contractor constants $(\alpha,\alpha_1,\alpha_2)$.

Then, if $k\alpha_i + \sum_j k_{1ij}\alpha_{1ij} + \sum_{j,\ell} k_{2ij\ell}\alpha_{2ij\ell} < 1$, there exists a solution to equation (1.1) in D^n.

Proof. Consider the sequence $\{x^{(m)}\}$ defined by

$$\begin{aligned} x_i^{(m+1)}(t;\omega) = x_i^{(m)}(t;\omega) - [y_i^{(m)}(t;\omega) &+ (T\Gamma y_i^{(m)})(t;\omega) \\ &+ \sum_j (T_{1ij}\Gamma_{1ij}\, y_i^{(m)})(t;\omega) \\ &+ \sum_{j,\ell} (T_{2ij\ell}\Gamma_{2ij\ell}\, y_i^{(m)})(t;\omega), \\ &m = 0,1,2,\dots;\quad i = 1,2,\dots,n, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} y_i^{(m)}(t;\omega) = x_i^{(m)}(t;\omega) &- x_{i0}(\omega) \\ &- (Th(s,x^{(m)}(s;\omega);\omega))(t;\omega) \end{aligned}$$

$$- \Big(\sum_j T_{1ij} f_{1ij}(s, x^{(m)}(s;\omega))\Big)(t;\omega)$$

$$- \Big(\sum_{j,\ell} T_{2ij\ell} f_{2ij\ell}(s, x^{(m)}(s;\omega))\Big)(t;\omega), \qquad (3.2)$$

and

$$x^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}), \ x_0(\omega) \in D^n.$$

It is clear from the definition of the operators Γ, Γ_{1ij}, and $\Gamma_{2ij\ell}$ and from the admissibility of (B,D) with respect to T, T_{1ij}, and $T_{2ij\ell}$ that $x^{(m)}$, $y^{(m)} \in D^n$. We shall now show that $||y_i^{(m)}||_D \to 0$ as $m \to \infty$. From equations (3.1) and (3.2), we have

$$y_i^{(m+1)}(t;\omega) = (Th(s, x^{(m)}(s;\omega);\omega))(t;\omega)$$

$$+ \sum (T_{1ij} f_{1ij}(s, x^{(m)}(s;\omega)))(t;\omega)$$

$$+ \sum_{j,\ell} (T_{2ij\ell} f_{2ij\ell}(s, x^{(m)}(s;\omega))(t;\omega)$$

$$- (T\Gamma y_i^{(m)})(t;\omega)$$

$$- \sum_j (T_{1ij} \Gamma_{1ij} y_i^{(m)})(t;\omega)$$

$$- \sum_{j,\ell} (T_{2ij\ell} \Gamma_{2ij\ell} y_i^{(m)})(t;\omega)$$

$$- Th(s, x^{(m)}(s;\omega) - y^{(m)}(s;\omega)$$

$$- (T\Gamma y^{(m)})(s;\omega) - [\sum_j (T_{1ij} \Gamma_{1ij} y^{(m)})](s;\omega)$$

$$- [\sum_{j,\ell} T_{2ij\ell} \Gamma_{2ij\ell} y^{(m)}](s;\omega);\omega)$$

$$- \sum_j T_{1ij} f_{1ij}(s, x^{(m)}(s;\omega) - y^{(m)}(s;\omega)$$

$$- (T\Gamma y^{(m)})(s;\omega) - [\sum_j T_{1ij}\Gamma_{1ij}y^{(m)}]$$

$$- [\sum_{j,\ell} T_{2ij\ell}\Gamma_{2ij\ell}y^{(m)}](s;\omega))(s;\omega)$$

$$- \sum_{j,\ell} T_{2ij\ell}f_{2ij\ell}(s,x^{(m)}(s;\omega) - y^{(m)}(s;\omega)$$

$$- (T\Gamma y^{(m)})(s;\omega) - [\sum_j T_{1ij}\Gamma_{1ij}y^{(m)}](s;\omega)$$

$$- (\sum_{j,\ell} T_{2ij\ell}\Gamma_{2ij\ell}y^{(m)})(s;\omega). \tag{3.3}$$

Equation (3.3) together with equations (2.8) and (2.9) enable us to conclude

$$||y_i^{(m+1)}||_D \le [k\alpha_i + \sum_j k_{1ij}\alpha_{1ij}$$

$$+ \sum_j k_{2ij\ell}\alpha_{2ij\ell}]||y_i||_D \le \alpha_i^*||y_i^{(m)}||D, \tag{3.4}$$

where $\alpha_i^* < 1$. It therefore follows that $||y_i^{(m)}||_D \to 0$ as $m \to \infty$. Also from equation (2.1) we have

$$||x_i^{(m+1)} - x_i^{(m)}||_D \le ||y_i^{(m)}||_D + k||\Gamma y_i^{(m)}||_B$$

$$+ \sum_j k_{1ij}||\Gamma_{1ij}y_i^{(m)}||_B$$

$$+ \sum_{j,\ell} k_{2ij\ell}||\Gamma_{2ij\ell}y_i^{(m)}||_B$$

$$\le [1 + Q^*k + Q_1^* \sum_j k_{1ij}$$

$$+ Q_2^* \sum_{j,\ell} k_{1ij\ell}]||y_i^{(m)}||_D \tag{3.5}$$

where Q^*, Q_i^*, $i = 1,2$ are upper bounds on $Q(t)$, $Q_i(t)$, $i = 1,2$.

It follows from (3.4) and (3.5) that the sequence $\{x_i^{(m)}\}$ is a Cauchy sequence in D. Hence, there exists an $x_i \in D$ such that $\lim_{m\to\infty} x_i^{(m)} = x_i$. In view of the continuity of operators and the functions involved in equation (3.2), it follows that x_i is a solution of equation (1.1) and this completes the proof. ///

We next prove a theorem concerning the uniqueness of solutions of equation (1.1).

Theorem 3.2. Let the hypotheses of Theorem 3.2 be satisfied. Assume in addition that the linear integral system

$$\begin{aligned}\eta_i(t;\omega) &= y_i(t;\omega)\\ &+ \sum_j \int_0^t r_{1ij}(t,\tau) \int_0^\tau K_{1ij}(t,s;\omega)(\Gamma_{1ij}(s,x(s;\omega))y_i(s,\omega)dz_j(s;\omega)d\tau\\ &+ \sum_{j,\ell} \int_0^t r_{2ij\ell}(t,\tau) \int_0^\tau K_{2ij\ell}(t,s;\omega)\Gamma_{2ij\ell}(s,x(s;\omega))y_i(s;\omega)dz_j(s;\omega)dz_\ell(s;\omega)d\tau\end{aligned} \tag{3.6}$$

has a solution $y_i(t;\omega)$ in D for every $x(t;\omega) \in D^n$ and $\eta_i(t;\omega) \in D$. Then equation (1.1) has a unique solution $x(t;\omega) \in D^n$.

Proof. Let $x_i^{(1)}(t;\omega)$ and $x_i^{(2)}(t;\omega)$ be two solutions in D of (1.1) corresponding to two initial random variables $x_{i,0}^{(1)}(\omega)$ and $x_{i,0}^{(2)}(\omega)$, $i = 1,2,\ldots,n$. Then

$$\begin{aligned}x_i^{(1)}(t;\omega) - x_i^{(2)}(t;\omega) &= x_{i,0}^{(1)}(\omega) - x_{i,0}^{(2)}(\omega) + [\int_0^t h_i(s,x^{(1)}(s;\omega);\omega)\\ &\quad - h_i(s,x^{(2)}(s;\omega);\omega)]ds\\ &+ \sum_j \int_0^t r_{1ij}(t,\tau) \int_0^\tau K_{1ij}(\tau,s;\omega)[f_{1ij}(s,x^{(1)}(s;\omega))\\ &\quad - f_{1ij}(s,x^{(2)}(s;\omega))\, dz_j(s;\omega)d\tau\end{aligned}$$

$$+ \sum_{i,j} \int_0^t r_{ij\ell}(t,\tau) \int_0^\tau K_{2ij\ell}(\tau,s;\omega)[f_{2ij\ell}(s,x^{(1)}(s;\omega))$$

$$- f_{2ij\ell}(s,x^{(2)}(s;\omega))]dz_j(s;\omega)dz_\ell(s;\omega)d\tau$$

$$i = 1,2,\ldots,n. \tag{3.7}$$

Letting $\eta_i(s;\omega) = x_i^{(1)}(t;\omega) - x_i^{(2)}(t;\omega)$ we can conclude from the fact that (3.6) has a solution for every z and x, that there is a $y_i(t;\omega) \in D$ such that

$$x_i^{(1)}(t;\omega) = x_i^{(2)}(t;\omega) + y_i(t;\omega)$$

$$+ \sum_j r_{1ij}(t,\tau) \int_0^\tau K_{1ij}(\tau,s;\omega)(\Gamma_{1ij}(s,x^{(2)}(s;\omega)y_i)(s;\omega)dz_j(s;\omega)d\tau$$

$$+ \sum_{j,\ell} r_{2ij\ell}(t,\tau) \int_0^\tau K_{2ij}(\tau,s;\omega)(\Gamma_{2ij\ell}(s,x^{(2)}(s;\omega))y_i)(s;\omega)dz_j(s;\omega)dz_\ell(s;\omega)d\tau. \tag{3.8}$$

From (3.7) and (3.8), and upon using the contractor condition and simplifying, we obtain

$$||y_i(t;\omega)||_D \le ||x_{i,0}^{(1)}(\omega) - x_{i,0}^{(2)}(\omega)||_D$$

$$+ (\alpha k + \sum_j \alpha_{1ij}k_{1ij} + \sum_{j,\ell} \alpha_{2ij\ell}k_{2ij\ell})||y_i(t;\omega)||_D.$$

from which we get

$$||y_i||_D \le (1 - \alpha k - \sum_j \alpha_{1ij}k_{1ij} - \sum_{j,\ell} \alpha_{2ij\ell}k_{2ij\ell})^{-1} \times ||x_{i,0}^{(1)} - x_{i,0}^{(2)}||_D. \tag{3.9}$$

If $x_{i,0}^{(1)} = x_{i,0}^{(2)}$, it follows that $y_i \equiv 0$, which in view of (3.8) implies that $x_i^{(1)} = x_i^{(2)}$. ///

We now consider the following special case of the equation (1.1) because of its usefulness in applications:

$$x(t;\omega) = h(t;\omega) + \int_0^t r_0(t,\tau) \int_0^\tau K_0(\tau,u;\omega) f_0(u,x(u;\omega))du\, d\tau$$
$$+ \sum_j \int_0^t r_{1j}(t,\tau) \int_0^\tau K_1(\tau,u;\omega) f_{1j}(u,x(u;\omega)) dz_j(u;\omega) d\tau$$
$$+ \sum_{j,\ell} \int_0^t r_{2j\ell}(t,\tau) \int_0^\tau K_2(\tau,u;\omega) f_{2j\ell}(u,x(u;\omega)) dz_j(u;\omega) dz_\ell(u;\omega) d\tau. \tag{3.10}$$

Let T_0 be the operator

$$(T_0x)(t;\omega) = \int_0^t r_0(t,\tau) \int_0^\tau K_0(\tau,u;\omega) x_0(u;\omega) du d\tau,$$

and let T_{1j} and $T_{2j\ell}$ be as defined in equations (2.6) and (2.7), respectively, with $i = 1$, $r_{1ij} = r_{1j}, r_{2ij\ell} = r_{2j\ell}$, $K_{1ij} = K_{1j}$ and $K_{2ij\ell} = K_{2j\ell}$. Also, let C_1^* be a subspace of the Banach space C_1.

<u>Theorem 3.3</u>. Assume the following:

(i) $h(t;\omega) \in C_1^*$.

(ii) The pair (C_1, C_1^*) is admissible with respect to each of the operators T_0, T_{1j}, and $T_{2j\ell}$.

(iii) There are positive constants c_0, c_{1j} and $c_{2j\ell}$ such that the operator $(I - c_0T_0)$ is a.s. invertible on C_∞ and the operator $(I - (I - c_0T_0)^{-1}(\sum_j c_{1j}T_{1j} + \sum_{j,\ell} c_{2j\ell}T_{2j\ell}))$ is invertible as a map from C_1 to C_1^*.

(iv) $|f_0(t,u+v) - f_0(t,u) - c_0v| \le \gamma_0|v|$,
$|f_{1j}(t,u+v) - f_{1j}(t,u) - c_{1j}v| \le \gamma_{1j}|v|$, and
$|f_{2j\ell}(t,u+v) - f_{2j\ell}(t,u) - c_{2j\ell}v| \le \gamma_{2j\ell}|v|$
for $u,v \in R$ where γ_0, γ_{1j}, and $\gamma_{2j\ell}$ are positive constants.

Then, if

$$||(I - (I - c_0T_0)^{-1}(\sum_j c_{1j}T_{1j} + \sum_{j,\ell} c_{2j\ell}T_{2j\ell}))^{-1}||$$

$$[\gamma_0||(I - c_0T_0)^{-1} T_0|| + \sum_j \gamma_{1j}||(I - c_0T_0)^{-1} T_{1j}||$$

$$+ \sum_{j,\ell} \gamma_{2j\ell}||(I - c_0T_0)^{-1}T_{2j\ell}|| < 1,$$

the equation (3.9) has a unique solution in C_1^* such that

$$\sup_{t\geq 0} ||x(t;\omega)||_2 \leq M \sup_{t\geq 0} ||h(t;\omega)||_2.$$

Proof. Using the invertibility of the operator $(I - c_0T_0)$, equation (3.10) can be written as

$$x(t;\omega) = \bar{h}(t;\omega) + \bar{T}_0\bar{f}_0(t,x(t;\omega))$$

$$+ \sum_j \bar{T}_{1j}f_{1j}(t,x(t;\omega))$$

$$+ \sum_{j,\ell} \bar{T}_{2j\ell}f_{2j\ell}(t,x(t;\omega)), \tag{3.11}$$

where

$$\bar{h}(t,\omega) = (I - c_0T_0)^{-1}h(t;\omega),$$

$$\bar{f}_0(t,x) = f_0(t,x) - c_0x,$$

$$\bar{T}_{1j} = (I - c_0T_0)^{-1} T_{1j},$$

and

$$\bar{T}_{2j\ell} = (I - c_0T_0)^{-1} T_{2j\ell}.$$

From condition (iv) of this theorem, it is easily verified that with the choice

$$\Gamma_0 = 0$$

$$\Gamma_{1j} = c_{1j}[I - \sum_j c_{1j}\,\bar{T}_{1j} - \sum c_{2j\ell}\bar{T}_{2j\ell}]^{-1}$$

$$\Gamma_{2j\ell} = c_{2j\ell}\,[I - \sum_j c_{1j}\,\bar{T}_{1j} - \sum c_{2j\ell}\bar{T}_{2j\ell}]^{-1}, \tag{3.12}$$

the set of functions $(f_0, f_{1j}, f_{2j\ell})$ has a vector of contractor constants $\alpha_0 = \gamma\,\gamma_0$, $\alpha_{1j} = \gamma\,\gamma_{1j}$, and $\alpha_{2j\ell} = \gamma\,\gamma_{2j\ell}$, where

$$\gamma = ||(I - \sum_j c_{1j}\bar{T}_{1j} - \sum_{j,\ell} c_{2j\ell}\,\bar{T}_{2j\ell})^{-1}||.$$

The existence of a solution now follows from Theorem 3.1.

To prove the uniqueness, we note that for the choice of contractors given by equations (3.12), the equality

$$I - \sum_j \bar{T}_{1j}\Gamma_{1j} - \sum_{j,\ell} \bar{T}_{2j\ell}\Gamma_{2j\ell} = I - \sum_j c_{1j}\bar{T}_{1j} - \sum_{j,\ell} c_{2j\ell}\,\bar{T}_{2j\ell}, \tag{3.13}$$

holds. Thus, the invertibility of the operator $(I - \sum_j c_{1j}\bar{T}_{1j} - \sum_{j,\ell} c_{2j\ell}\,\bar{T}_{2j\ell})$ implies that the equation

$$\eta(t;\omega) = (I - \sum_j c_{1j}\,\bar{T}_{1j} - \sum_{j,\ell} c_{2j\ell}\,\bar{T}_{2j\ell})y(t;\omega)$$

has a unique solution for each $\eta \in C_1^*$. Therefore, by Theorem 3.2 and from equation (3.9) the solution is unique and satisfies the condition

$$\sup_{t\ge 0} ||x(t;\omega)||_2 \le M \sup_{t\ge 0} ||h(t;\omega)||_2.$$

///

4. Application to the Stability of Stochastic Systems

In this section we shall discuss the stability of solutions of a special

class of feedback systems with convolution kernels, sector-bounded nonlinearities, and disturbances comprised of a Brownian motion and a Poisson point process. Using the contractor theory, we shall obtain sufficient conditions under which the feedback system is mean-square stable [3]. The conditions involve the sector parameters of the nonlinearities, the parameters of the stochastic disturbances and the Laplace transforms of the kernels.

Consider the feedback system

$$\begin{aligned} x(t;\omega) = h(t;\omega) &- \int_0^t r_0(t-\tau) \int_0^\tau K_0(\tau-u) f_0(u,x(u;\omega))du\, d\tau \\ &- \int_0^t r_1(t-\tau) \int_0^\tau K_1(\tau-u) f_1(u,x(u;\omega))dz_1(u;\omega)d\tau \\ &- \int_0^t r_2(t-\tau) \int_0^\tau K_2(\tau-u) f_2(u,s(u,\omega))dz_2(u;\omega)d\tau. \qquad (4.1) \end{aligned}$$

The process $z_1(t;\omega)$ and $z_2(t;\omega)$ in (4.1) satisfy:

(i) $z_1(t;\omega)$ is a random process such that $z_1(0;\omega) = 0$ and

$$z(t;\omega) - z_1(s;\omega) = \sum_{s<\tau\le t} v(\tau;\omega)(N(\tau;\omega) - N(\tau^-;\omega)).$$

The random function $v(\tau;\omega)$ has mean μ and variance σ^2 and, corresponding to any finite set $\tau_1,\ldots,\tau_k$, the random variables $v(\tau_i;\omega)$ are independent, and $N(t;\omega)$ is a Poisson process with parameter λ (see [5], page 88). Furthermore, $v(t;\omega)$ is independent of all increments $N(t;\omega) - N(s;\omega)$.

(ii) $z_2(t;\omega)$ is a standard Brownian motion process.

Under the above assumptions the following results are known [5]:

$$E \int_0^t f(t,\tau)dz_2(\tau;\omega) = 0;\ E(\int_0^t f(t,\tau)dz_2(\tau;\omega))^2 = \int_0^t E\, f^2(t,\tau)d\tau; \qquad (4.2)$$

$$E \int_0^t f(t,\tau)dz_1(\tau;\omega) = \lambda\mu \int_0^\tau f(t,\tau)d\tau ; \tag{4.3}$$

$$E\{\int_0^t f(t,\tau)d[z_1(\tau;\omega) - \mu\lambda\tau]\}^2 = \int_0^t E\, f^2(t,\tau)\lambda(\mu^2 + \sigma^2)d\tau. \tag{4.4}$$

We shall further assume that z_1 and z_2 are independent and h is admissible and non-anticipating [3]. The functions r_i, K_i, and $f_i (i = 0,1,2)$ are all continuous. The nonlinearities f_i are sector-bounded and satisfy the condition

$$|f_i(t,u + v) - f_i(u) - c\ v| \le \frac{b-a}{2} |v|$$

where $c = \frac{a+b}{2}$ $(i = 0,1,2)$, $u,v \in R$.

In what follows, we shall denote the Laplace transform of a function $r(t)$ by $\hat{r}(s)$ and the convolution of two functions f and g by $f * g$. We will now present a stability theorem for the system (4.1).

Theorem 4.1 Assume

(i) $(-\frac{1}{c}, i\,0) \notin \bigcup_{\mathrm{Re}\, s \ge -\beta_0} \hat{R}_0(s)$

where $\hat{R}_0(s) = [\hat{r}_o(s)\ \hat{K}_0(s) + \mu\lambda\hat{r}_1(s)\hat{K}_1(s)]$

and $\beta_0 > 0$ is such that

$$\int_0^\infty e^{-\beta_0 t} (|r_0(t)K_0(t)| + |r_1(t)K_1(t)|)dt < \infty.$$

(ii) Let $\hat{R}_i(s) = (1 - \hat{R}_0(s))^{-1}\hat{r}_i(s)$, $i = 1,2$ and

$$\hat{W}(s) = \lambda(\mu^2 + \sigma^2)[(\hat{R}_1(s)\hat{K}_1(s))*(\hat{R}_1(s)\hat{K}_1(s))]$$
$$(\hat{R}_1(s)\hat{K}_2(s))*(\hat{R}_1(s)\hat{K}_2(s)),$$

so that $\left(-\frac{2}{a^2+b^2}, i\,0\right) \notin \bigcup_{\mathrm{Re}\, s \ge -\beta_0} \hat{W}(s).$

(iii) $$\sup_{s\geq-\beta_0}\left|\left(1-\frac{a^2+b^2}{2}\hat{W}(s)\right)^{-1}\right|$$

$$\times\left\{\sup_{\xi\in R}\left|\int_{-\infty}^{+\infty}\frac{\hat{r}^*(\mu+j(\xi-\xi_0))\hat{r}^*(\mu+j\xi_0)}{(\mu+j(\xi-\xi_0))(\mu+j\xi_0)}d\xi_0\right|\right.$$

$$\left.+\sup_{s\geq-\beta_0}W(s)\right\}<\frac{2}{b-a}$$

where $\hat{r}^*(s) = (1-\hat{R}_0(s))^{-1}[\hat{r}_0(s)\,\hat{K}_0(s)+\hat{r}_1(s)\,\hat{K}_1(s)]$

and $$\sup_{s\geq-\beta_0}\hat{W}(s) > \frac{2}{a^2+b^2}\,.$$

(iv) $h \in C_1$.

Then there exists a unique solution $x(t;\omega)$ of (4.1) such that

$$\sup_{t\geq0}||x(t;\omega)||_2 \leq M\sup_{t\geq0}||h(t;\omega)||_2$$

for some $M > 0$.

Before we proceed with the proof of the theorem, we shall first prove the following lemma.

Lemma 4.1. Let T_1^* and T_2^* be operators on C_∞ defined by

$$T_1^*\,v(t) = \int_0^t (r_1 * K_1)^2(t-\tau)v(\tau)d\tau, \qquad (4.5)$$

$$T_2^*\,v(t) = \int_0^t (r_2 * K_2)^2(t-\tau)v(\tau)d\tau. \qquad (4.6)$$

Let T_1 and T_2 be operators on C_1 defined by

$$T_1x(t;\omega) = \int_0^t r_1(t-\tau) \int_0^\tau K_1(\tau-u)x(u;\omega)d(z_1(u;\omega) - \mu\lambda)du, \quad (4.7)$$

$$T_2x(t;\omega) = \int_0^t r_2(t-\tau) \int_0^\tau K_2(\tau-u)x(u;\omega)dz_2(u;\omega)d\tau, \quad (4.8)$$

where r_1, r_2, K_1, K_2, z_1 and z_2 are as in equation (4.1). Assume that there exist constants $c_1, c_2 > 0$ such that $||c_1T_1 + c_2T_2|| > 1$ and $(I - c_1^2 T_1^* - c_2^2 T_2^*)$ is invertible. Then the operator $(I - c_1T_1 - c_2T_2)$ is invertible as a map from C_1 to C_1^* (C_1^* = Range of $T_1 + T_2$) and

$$||(I - c_1T_1 - c_2T_2)^{-1}||^2 \le ||(I - c_1^2 T_1^* - c_2^2 T_2^*)^{-1}||.$$

<u>Proof</u>. The assumption that $||c_1T_1 + c_2T_2|| > 1$ implies that for some $\lambda(t) > 1$, there is a $y^* \in C_1$ such that

$$||(c_1T_1 + c_2T_2)y^*||_2 = \lambda(t)||y^*||_2. \quad (4.9)$$

Let $\quad A \equiv \{y^* \mid\, ||(c_1T_1 + c_2T_2)y^*||_2 = \lambda(t)||y^*||_2\}. \quad (4.10)$

Let $\quad x,y \in C_1$ such that $y = x + c_1T_1y + c_2T_2y. \quad (4.11)$

Then

$$||y||_2^2 \le ||x||_2^2 - ||c_1T_1y||_2^2 - ||c_2T_2y||_2^2 + 2||y||_2[||c_1T_1y + c_2T_2y||_2]. \quad (4.12)$$

Choose $y^* \in A$ such that $||y^*||_2 = ||y||_2$. Then

$$\begin{aligned} ||y||_2^2 = ||y^*||_2^2 &\le ||x||_2^2 - ||c_1T_1y^*||_2^2 - ||c_2T_2y^*||_2^2 \\ &\quad + 2||c_1T_1y^* + c_2T_2y^*||^2[\lambda(t)]^{-1} \\ &\le ||x||_2^2 + (c_1^2 T_1^* + c_2^2 T_2^*)||y||_2^2. \end{aligned} \quad (4.13)$$

Hence,

$$||y||_2^2 \le ||(I - c_1^2T_1^* - c_2^2T_2^*)^{-1}|| \quad ||x||_2^2 . \tag{4.14}$$

Equations (4.11) and (4.14) allow us to conclude that

$$||(I - c_1T_1 - C_2T_2)^{-1}||^2 \le ||(I - c_1^2T_1^* - c_2^2T_2^*)^{-1}||.$$

To show that $(I - c_1T_1 - c_2T_2)$ has an inverse, we note that the operator is an onto map from C_1 to C_1^*. Therefore, we have only shown that it is one-to-one. Let $y \in C_1^*$ such that $y = (1 - c_1T_1 - c_2T_2)x_1 = (1 - c_1T_1 - c_2T_2)x_2$ for some $x_1 \neq x_2$. Then

$$E(x_1-x_2)^2 - [c_1^2T_1^* + c_2^2T_2^*]\; E(x_1-x_2)^2 = 0. \tag{4.15}$$

From the invertibility of the operator $(I - c_1^2T_1^* - c_2^2T_2^*)$ it follows that $x_1 = x_2$ almost surely, which shows that the map $(I - c_1T_1 - c_2T_2)$ has an inverse. ///

<u>Proof of Theorem 4.1</u>. Rewrite equation (4.1) as

$$x(t;\omega) = \bar{h}(t;\omega) + \sum_{i=0}^{1} \bar{T}_i\bar{f}_i(t,x(t;\omega)) + \sum_{i=2}^{3} \bar{T}_i f_i(t,x(t;\omega)), \tag{4.16}$$

where

$$\bar{h} = (I - c\,T_0 - c\,T_1)^{-1}h,$$

$$\bar{T}_i = (I - c\,T_0 - c\,T_1)^{-1}T_i\,, \qquad i = 0,1,2,3$$

$$\bar{f}_i(t,x) = f_i(t,x) - c\,x\,, \qquad i = 0,1, \text{ and } c = \frac{a+b}{2} \tag{4.17}$$

The operators T_0, T_2 and T_3 are defined as in Theorem 3.3 (with kernels being convolution) and T_1 is defined by

$$T_1x(t;\omega) = \int_0^t r_1(t-\tau) \int_0^\tau K_1(\tau-u)x(u;\omega)\mu\lambda \, du d\tau. \tag{4.18}$$

The existence of $(I - cT_0 - cT_1)^{-1}$ follows from assumption (i) of the theorem [3].

We shall now show that the assumptions of Theorem 4.1 imply those of Theorem 3.3. Condition (ii) of the theorem implies that the inverse of $(I - c^2 \bar{\bar{T}}_1{}^* - c^2 \bar{\bar{T}}_2{}^*)$ exists [3]. Therefore, from Lemma 4.1 it follows that the operator $(I - c(I - c(T_0+T_1))^{-1}(T_1+T_2))$ is invertible. Further, condition (iii) [3] and Lemma 4.1 together imply that

$$\begin{aligned} \frac{(b-a)}{2} & \|(I - c(I - c(T_0+T_1))^{-1}(T_1+T_2))^{-1}\| \\ & \times [\|(I - c(T_0+T_1))^{-1}(T_0+T_1)\| \\ & \qquad + \|(I - c(T_0+T_1))^{-1}(T_1+T_2)\|] < 1 \end{aligned}$$

The other conditions are easily verified and the conclusion follows from Theorem 3.3. ///

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